

PROOF OF THE MODULAR BRANCHING RULE FOR CYCLOTOMIC HECKE ALGEBRAS

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ABSTRACT. We prove the modular branching rule of the cyclotomic Hecke algebras, which has remained open.

1. INTRODUCTION

Let F be an algebraically closed field. The cyclotomic Hecke algebra $\mathcal{H}_n = \mathcal{H}_n(\underline{v}, q)$ of type $G(m, 1, n)$ is the F -algebra introduced in [AK] and [BM]. This is a cellular algebra in the sense of Graham and Lehrer, and the cell module theory of this algebra is nothing but the Specht module theory developed by Dipper, James and Mathas [DJM1].¹ The Specht modules are parametrized by m -tuples of partitions $\lambda = (\lambda^{(1)}, \dots, \lambda^{(m)})$ and denoted by S^λ . Each S^λ has an invariant symmetric bilinear form, and we denote by D^λ the module obtained from S^λ by factoring out the radical of the invariant form. Then nonzero D^λ 's form a complete set of irreducible \mathcal{H}_n -modules.

If we set $m = 1$, \mathcal{H}_n is the Hecke algebra of type A . If we further set $q = 1$, then \mathcal{H}_n is the group algebra of the symmetric group S_n . Kleshchev studied $\text{Soc}(\text{Res}_{S_{n-1}}^{S_n}(D^\lambda))$ in a series of papers [Kl1] to [Kl4], and obtained an explicit rule for describing the socle. This is called the modular branching rule of the symmetric group. The method is to use modified lowering operators, and Brundan generalized this result to the Hecke algebra of type A by the same method [B].

Around the same time, motivated by conjectures and results by Lascoux, Leclerc and Thibon, a link between quantum groups of type $A_{e-1}^{(1)}$ and the Hecke algebra of type A was found. In particular, they observed that the crystal rule of Misra and Miwa coincides with Kleshchev's rule for the modular branching [LLT].

On the other hand, in solving the LLT conjecture on the decomposition numbers, I generalized the LLT conjecture to the graded dual of Grothendieck groups of the module categories of \mathcal{H}_n with common parameters. With this interpretation, the action of Chevalley generators is given by refined restriction and induction functors, which are the i -restriction and i -induction functors.² Further, by using Lusztig's canonical basis in the proof, it was natural for us to observe the existence of a crystal structure on the set

$$B = \bigsqcup_{n \geq 0} \{\text{isoclasses of simple } \mathcal{H}_n\text{-modules}\}.$$

¹Specht module theory for Hecke algebras was initiated by Dipper and James, and the Specht module theory we use here is its generalization to the cyclotomic Hecke algebras.

²The use of central elements in the symmetric group goes back to Robinson, which I learned from Leclerc, but the refined induction and restriction operators in this context were introduced by the author.

³ In this theory, which we call Fock space theory, we may identify the crystal with \mathcal{KP} of those multipartitions for which $D^\lambda \neq 0$. ⁴ Its rigidity, namely independence of the characteristic of F , was first proved in [AM]. The crystal is isomorphic to the $\mathfrak{g}(A_{e-1}^{(1)})$ -crystal of an integrable highest weight module $L_v(\Lambda)$, where e is the multiplicative order of the parameter $q \neq 1$ and Λ is determined by the parameters \underline{v} . For the overview of the Fock space theory, see [Abook].

As many people in our field noticed, these works give a natural conjecture generalizing the results of Kleshchev and Brundan on modular branching rules for the symmetric groups and the Hecke algebras of type A; that is, we have a natural conjecture for a modular branching rule for the cyclotomic Hecke algebras. Explicitly, this asserts that $\text{Soc}(e_i D^\lambda) = D^{\tilde{e}_i \lambda}$, where e_i is the i -restriction and \tilde{e}_i is the Kashiwara operator of the crystal \mathcal{KP} .

There was a progress toward this conjecture in Vazirani's thesis, which was later published as [GV]. In the thesis, various facts which are necessary to show that B has a crystal structure are proven, and they are used in [G] to show that our B is equipped with another crystal structure. This crystal structure is again isomorphic to the crystal of the same integrable highest module [G, Theorem 14.3]. In fact, the proof is carried out within the framework of my Fock space theory.

On this occasion, I correct two of his announcements which are relevant to the modular branching rule, as service to the mathematical community and to avoid confusions. In [GV], it is said: "What we do not do in this paper is to explicitly describe which irreducible representations occur in the socle of the restriction. This is done in [G], generalizing [Kv](=Kleshchev's work) which describes the combinatorics of the branching rule for the symmetric group explicitly in terms of p -regular partitions." However, in [G] one only finds such a result in terms of an abstractly defined crystal graph, and no attempt is made to give an explicit description of the latter in terms of partitions. Moreover, Grojnowski left completely untouched the problem of matching up the standard labeling of simple modules coming from Specht module theory with his labeling coming from the abstract crystal graph. So, contrary to the announcement recorded in the note added in proof of Mathas' book [Mbook, p.135], no proof of the modular branching rule (even in the case of Brundan and Kleshchev's original modular branching rule) is present in [GV] or in [G] (except in the case where q is not a root of unity which was treated in [V]).

As modular branching is used as the definition of their crystal, it is more appropriate to see their theory as a method to label simple modules using a crystal, rather than as a modular branching rule. ⁵ The adjoint operation to modular branching is to take head of induction, and they use this as the method to label simple \mathcal{H}_n -modules. This means that we need to repeat the operation of taking the head of an induced module n times to compute a simple \mathcal{H}_n -module this way.

Let us examine in more detail how to compute the label of a given module, and modular branching, by this method. Suppose that we are given a simple \mathcal{H}_n -module V and that we have computed its character, namely its restriction to the commutative subalgebra generated by Jucy-Murphy elements. Then we can

³This was already mentioned in the form of its relationship with Kashiwara's lower crystal basis in [A1, p.807].

⁴We named these multipartitions Kleshchev multipartitions in [AM].

⁵In fact, viewing the theory this way, Brundan and Kleshchev were able to label simple modules of the Hecke-Clifford algebra by using $\mathfrak{g}(A_{2l}^{(2)})$ -crystal.

compute the character of $e_i V$. To know $\text{Soc}(e_i V)$, we have to rewrite the character into summation over characters of simple \mathcal{H}_{n-1} -modules V' and compute the values $\epsilon_i(V')$. Thus we are required to know the irreducible characters.⁶ The only way to compute the irreducible characters in the method is to construct the modules by taking head of induction as above.⁷ One can compute the character of an induced module, but we meet the same problem for computing its head. Thus, to compute the labeling or modular branching, the only way is to compute socle of restriction (or head of induction) explicitly.

Finding the label λ of a given module in Specht module theory is also not automatic, but we have more realistic chance for finding the label. For example, λ is the minimal Kleshchev multipartition that satisfies $\text{Hom}_{\mathcal{H}_n}(S^\lambda, V) \neq 0$. Further, the original modular branching rule allows us to compute the socle of the restriction without computing the socle. It is also worth mentioning that our approach of using the Specht module theory is still the only alternative even for proving Brundan's result in type A. Thus, the importance of the Specht module theory could not be overestimated.

The purpose of this paper is to prove the modular branching rule of cyclotomic Hecke algebras, which until now has remained open. It turns out that it is a direct consequence of the theorem on the canonical basis in the Fock space. [GV] and [G] contain new results also and we use two of them in the proof.⁸

2. PRELIMINARIES

Definition 2.1. Let R be a commutative ring, and let $v_1, \dots, v_m, q \in R$ be invertible elements. The cyclotomic Hecke algebra $\mathcal{H}_n(\underline{v}, q)$ is the R -algebra defined by the generators T_0, \dots, T_{n-1} and the relations

$$(T_0 - v_1) \cdots (T_0 - v_m) = 0, \quad (T_i - q)(T_i + 1) = 0, \text{ for } i \geq 1,$$

$$(T_0 T_1)^2 = (T_1 T_0)^2,$$

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \text{ for } i \geq 1,$$

$$T_i T_j = T_j T_i, \text{ for } j \geq i + 2.$$

We write \mathcal{H}_n for short. It is known that \mathcal{H}_n is free of rank $m^n n!$ as an R -module. We define elements L_1, \dots, L_n by

$$L_1 = T_0, \quad L_{k+1} = q^{-1} T_k L_k T_k, \text{ for } 1 \leq k < n.$$

They pairwise commute and the symmetric polynomials in L_1, \dots, L_n are central elements of \mathcal{H}_n .

⁶In modular representation theory, knowing irreducible characters is a hard problem. One may list the modular representation theory of the symmetric group, the Kazhdan-Lusztig and Lusztig conjectures, as examples. Note that knowing irreducible characters is equivalent to computing decomposition numbers.

⁷Here, Specht module theory provides us with easier way to construct the simple modules, but it is still unrealistic to compute the irreducible characters by constructing simple modules.

⁸When writing this paper, I learned that Brundan had a very similar idea for the proof. He considered a similar problem in a different setting [BK2, Theorem 4.4], and observed that the same strategy works in the present situation. I thank Brundan for the communication.

The Specht module theory for \mathcal{H}_n is developed by Dipper, James and Mathas [DJM1]. Recall that the set of multipartitions, namely the set of m -tuples of partitions, of size n is a poset whose partial order is the dominance order \supseteq . Let $\lambda = (\lambda^{(1)}, \dots, \lambda^{(m)})$ be a multipartition of size n . Then we can associate an \mathcal{H}_n -module S^λ with λ , called a Specht module. S^λ is free as an R -module. Further, each Specht module is equipped with an invariant symmetric bilinear form [DJM1, (3.28)]. Let $\text{rad } S^\lambda$ be the radical of the invariant symmetric bilinear form, and we set $D^\lambda = S^\lambda / \text{rad } S^\lambda$. We denote the projective cover of D^λ by P^λ when $D^\lambda \neq 0$.

Theorem 2.2 ([DJM1, Theorem 3.30]). *Suppose that R is a field. Then,*

- (1) *Nonzero D^λ form a complete set of non-isomorphic simple \mathcal{H}_n -modules. Further, these modules are absolutely irreducible.*
- (2) *Let λ and μ be multipartitions of size n and suppose that $D^\mu \neq 0$ and that $[S^\lambda : D^\mu] \neq 0$. Then $\lambda \supseteq \mu$. Further, $[S^\lambda : D^\lambda] = 1$.*

The projective cover P^μ has a Specht filtration

$$P^\mu = F_0 \supset F_1 \supset \dots$$

such that $F_0/F_1 \simeq S^\mu$. This follows from the cellularity of \mathcal{H}_n .

By the Morita-equivalence theorem of Dipper and Mathas [DM], we may assume that v_i are powers of q without loss of generality. In the rest of paper, we assume that q is a primitive e^{th} root of unity where $e \geq 2$, and $v_i = q^{\gamma_i}$, for $\gamma_i \in \mathbb{Z}/e\mathbb{Z}$.

3. THE KASHIWARA CRYSTAL

Let $A = (a_{ij})_{i,j \in I}$ be a generalized Cartan matrix, $\mathfrak{g} = \mathfrak{g}(A)$ the Kac-Moody Lie algebra associated with A . Let $(P, \Delta, P^\vee, \Delta^\vee)$ be the simply-connected root datum of \mathfrak{g} . We write α_i for simple roots, and h_i for simple coroots. Thus, P^\vee is generated by $\{h_i\}_{i \in I}$ and $|I| - \text{rank}(A)$ elements $\{d_s\}$ as a \mathbb{Z} -module.

Definition 3.1. A \mathfrak{g} -crystal B is a set endowed with

- $wt : B \longrightarrow P$,
- $\epsilon_i, \varphi_i : B \longrightarrow \mathbb{Z} \sqcup \{-\infty\}$,
- $\tilde{e}_i, \tilde{f}_i : B \longrightarrow B \sqcup \{0\}$,

such that the following properties are satisfied.

- (1) $\varphi_i(b) = \epsilon_i(b) + \langle h_i, wt(b) \rangle$.
- (2) If $b \in B$ is such that $\tilde{e}_i b \neq 0$ then

$$wt(\tilde{e}_i b) = wt(b) + \alpha_i, \quad \epsilon_i(\tilde{e}_i b) = \epsilon_i(b) - 1, \quad \varphi_i(\tilde{e}_i b) = \varphi_i(b) + 1.$$
- (3) If $b \in B$ is such that $\tilde{f}_i b \neq 0$ then

$$wt(\tilde{f}_i b) = wt(b) - \alpha_i, \quad \epsilon_i(\tilde{f}_i b) = \epsilon_i(b) + 1, \quad \varphi_i(\tilde{f}_i b) = \varphi_i(b) - 1.$$
- (4) For $b, b' \in B$, we have $b' = \tilde{e}_i b \iff \tilde{f}_i b' = b$.
- (5) If $b \in B$ is such that $\varphi_i(b) = -\infty$ then $\tilde{e}_i(b) = 0$ and $\tilde{f}_i(b) = 0$.

Let $U_v(\mathfrak{g})$ be the quantized enveloping algebra and $L_v(\Lambda)$ an integrable highest weight $U_v(\mathfrak{g})$ -module. Then the lower crystal base $B(\Lambda)$ of $L_v(\Lambda)$ is a \mathfrak{g} -crystal. Further, the crystal $B(\Lambda)$ is semiregular. That is,

$$\epsilon_i(b) = \max\{k \in \mathbb{Z}_{\geq 0} \mid \tilde{e}_i^k b \neq 0\}, \quad \varphi_i(b) = \max\{k \in \mathbb{Z}_{\geq 0} \mid \tilde{f}_i^k b \neq 0\}.$$

The module $L_v(\Lambda)$ has a distinguished basis, which is called Kashiwara's (lower) global basis or the Lusztig canonical basis. The basis elements are labelled by $B(\Lambda)$, and we denote them by $\{G_v(b)\}_{b \in B(\Lambda)}$. See [HK] for example.

The following lemma is taken from [K2, Lemma 12.1]. For the proof, follow the argument in [K1, Proposition 5.3.1] which is for the upper global basis.

Lemma 3.2. *Let $B(\Lambda)$ be the crystal of the integrable highest weight module $L_v(\Lambda)$. Then the following hold.*

- (1) *There exist Laurent polynomials $e_{bb'}^i(v)$ such that*

$$e_i G_v(b) = [\varphi_i(b) + 1] G_v(\tilde{e}_i b) + \sum_{b'} e_{bb'}^i(v) G_v(b'),$$

where the sum is over $b' \in B(\Lambda)$ with $\varphi_j(b') \geq \varphi_j(b) + \langle h_j, \alpha_i \rangle$, for all j .

- (2) *There exist Laurent polynomials $f_{bb'}^i(v)$ such that*

$$f_i G_v(b) = [\epsilon_i(b) + 1] G_v(\tilde{f}_i b) + \sum_{b'} f_{bb'}^i(v) G_v(b'),$$

where the sum is over $b' \in B(\Lambda)$ with $\epsilon_j(b') \geq \epsilon_j(b) + \langle h_j, \alpha_i \rangle$, for all j .

In this paper, we only use the affine Kac-Moody Lie algebra of type $A_{e-1}^{(1)}$, where e is defined by the parameter q as in the previous section. The crystal we use is the $A_{e-1}^{(1)}$ -crystal $B(\Lambda)$, where $\Lambda = \sum_{i=1}^m \Lambda_{\gamma_i}$ and γ_i are $v_i = q^{\gamma_i}$ as before.

4. FOCK SPACE THEORY

The Fock space theory is explained in detail in [Abook]. Let \mathfrak{g} be the affine Kac-Moody Lie algebra of type $A_{e-1}^{(1)}$. In [A1], I introduced the combinatorial Fock space $\mathcal{F}(\Lambda)$. It is a based \mathbb{Q} -vector space whose basis is the set of all multipartitions \mathcal{P} . The weight Λ defines a rule to color nodes of multipartitions with e colors $\mathbb{Z}/e\mathbb{Z}$, and the coloring rule defines an integrable \mathfrak{g} -module structure on $\mathcal{F}(\Lambda)$. Its deformation $\mathcal{F}_v(\Lambda)$ becomes an integrable $U_v(\mathfrak{g})$ -module via the Hayashi action, and the crystal obtained from $\mathcal{F}_v(\Lambda)$ is \mathcal{P} . Let $W_i(\lambda)$ be the number of i -nodes in λ . Then by the definition of the Hayashi action, we have

$$wt(\lambda)(h_i) = \Lambda(h_i) + W_{i-1}(\lambda) - 2W_i(\lambda) + W_{i+1}(\lambda), \text{ for } 0 \leq i \leq e-1,$$

$$wt(\lambda)(d) = \Lambda(d) - W_0(\lambda).$$

Recalling $\alpha_j(h_i) = a_{ij}$ and $\alpha_j(d) = \delta_{j0}$, this is equivalent to

$$wt(\lambda) = \Lambda - \sum_{j=0}^{e-1} W_j(\lambda) \alpha_j.$$

Kashiwara operators \tilde{e}_i and \tilde{f}_i are defined by removing or adding a good i -node. As \mathcal{P} is semiregular, ϵ_i and φ_i are determined by \tilde{e}_i and \tilde{f}_i . Then $(\mathcal{P}, \tilde{e}_i, \tilde{f}_i, wt, \epsilon_i, \varphi_i)$ is the crystal structure given on \mathcal{P} .

The connected component of \mathcal{P} that contains the empty multipartition \emptyset is denoted by \mathcal{KP} , and we call multipartitions in \mathcal{KP} Kleshchev multipartitions. The global basis $\{G_v(\lambda)\}_{\lambda \in \mathcal{KP}}$ is the basis of the $U_v(\mathfrak{g})$ -submodule generated by \emptyset , which is isomorphic to the irreducible highest weight $U_v(\mathfrak{g})$ -module $L_v(\Lambda)$. Similarly, the basis $\{G_v(\lambda)\}_{\lambda \in \mathcal{KP}}$ evaluated at $v = 1$ is the basis of the \mathfrak{g} -submodule generated by \emptyset , which is isomorphic to the irreducible highest weight \mathfrak{g} -module $L(\Lambda)$. We denote $\{G_v(\lambda)\}_{\lambda \in \mathcal{KP}}$ evaluated at $v = 1$ by $\{G(\lambda)\}_{\lambda \in \mathcal{KP}}$.

Suppose that the ground ring R of \mathcal{H}_n is an algebraically closed field F of characteristic ℓ , and recall that q is a primitive e^{th} root of unity and $v_i = q^{\gamma_i}$, for $\gamma_i \in \mathbb{Z}/e\mathbb{Z}$. Let $\mathcal{H}_n\text{-proj}$ be the category of (finite dimensional) projective \mathcal{H}_n -modules. In [A1], I defined the i -restriction and the i -induction functors. Let us recall the definitions following [Abook, 13.6]. Let M be an \mathcal{H}_n -module. As symmetric polynomials in L_1, \dots, L_n are central elements in \mathcal{H}_n , the simultaneous generalized eigenspace with respect to the symmetric polynomials in L_1, \dots, L_n is again an \mathcal{H}_n -module. Let $c = \{c_1, \dots, c_n\}$ where $c_i \in q^{\mathbb{Z}}$, and denote by $P_c(M)$ the simultaneous generalized eigenspace which consists of $m \in M$ such that

$$(f(L_1, \dots, L_n) - f(c_1, \dots, c_n))^N m = 0,$$

for $N \gg 0$ and for symmetric polynomials f . Define

$$e_i M = \sum_c P_{c \setminus \{q^i\}}(\text{Res}_{\mathcal{H}_{n-1}}^{\mathcal{H}_n}(P_c(M))), \quad f_i M = \sum_c P_{c \cup \{q^i\}}(\text{Ind}_{\mathcal{H}_{n-1}}^{\mathcal{H}_n}(P_c(M))).$$

e_i is the i -restriction and f_i is the i -induction. They are exact functors.

Suppose that R is a discrete valuation ring, K its fraction field, and M an \mathcal{H}_n -module which is torsionless as a R -module. Then we have $M \subset M \otimes_R K$, where $M \otimes_R K$ is an $\mathcal{H}_n \otimes_R K$ -module, and the definitions of e_i and f_i make sense for M . Further, $e_i(M \otimes_R K) = (e_i M) \otimes_R K$ and $f_i(M \otimes_R K) = (f_i M) \otimes_R K$ hold.

The following are main results of [A1]. See sections (4.5), (4.6), Theorem 4.4 and Proposition 4.5 in [A1], or Theorem 12.5 and Proposition 13.41 in [Abook].

Theorem 4.1. *Let $K_0(\mathcal{H}_n\text{-proj})$ be the Grothendieck group of $\mathcal{H}_n\text{-proj}$. Then*

- (1) *The action of e_i and f_i on $K(\Lambda) = \bigoplus_{n \geq 0} K_0(\mathcal{H}_n\text{-proj})$ satisfy the Serre relations, and extends to a \mathfrak{g} -module structure on $K(\Lambda)$.*
- (2) *$K(\Lambda)$ is isomorphic to the integrable \mathfrak{g} -module $L(\Lambda)$.*
- (3) *We have a unique injective \mathfrak{g} -module homomorphism $K(\Lambda) \rightarrow \mathcal{F}(\Lambda)$ which sends the highest weight vector $[P^\emptyset]$ to the empty multipartition \emptyset .*
- (4) *Assume that the characteristic of F is zero, and that $D^\lambda \neq 0$. Then $[P^\lambda]$ maps to a basis element $G(\lambda')$, for some $\lambda' \in \mathcal{KP}$, and we have*

$$G(\lambda') = \lambda + (\text{higher terms}) = \sum_{\mu \geq \lambda} d_{\mu\lambda} \mu,$$

where $d_{\mu\lambda} = [S^\mu : D^\lambda]$, the decomposition numbers.

Note that the existence of a crystal structure on the set

$$B = \bigsqcup_{n \geq 0} \{\text{isoclasses of simple } \mathcal{H}_n\text{-modules}\},$$

is clear from this theorem. That $\lambda' = \lambda$ is proved in [A2]. In particular, $D^\lambda \neq 0$ if and only if $\lambda \in \mathcal{KP}$ and we can identify B with \mathcal{KP} .

For each simple module D^λ , we have that any symmetric polynomial f in L_1, \dots, L_n acts as a scalar. Because of our assumption that v_i are powers of q , the eigenvalues of L_k , for $1 \leq k \leq n$, are powers of q . This is because they are powers of q for Specht modules. Thus, we have a uniquely determined set $\{q^{i_1}, \dots, q^{i_n}\}$ such that every symmetric polynomial $f(L_1, \dots, L_n)$ acts on D^λ as the scalar $f(q^{i_1}, \dots, q^{i_n})$. Observe that the symmetric polynomials act as scalars on S^λ already, and we can describe the set $\{q^{i_1}, \dots, q^{i_n}\}$ explicitly as follows.

$$|\{k \in [1, n] | q^{i_k} = q^i\}| = W_i(\lambda).$$

This module theoretic interpretation of $W_i(\lambda)$ was used in [A1], and will be used in the next section.

5. ANOTHER CRYSTAL STRUCTURE

Grojnowski and Vazirani introduced another semiregular crystal structure on the set B . The i -restriction they use [GV, 3.1] is precisely the one which I introduced in [A1]. f_i is left and right adjoint to e_i . As one can see from the definition of f_i given before, the definition is in terms of generalized eigenspace of L_n . Grojnowski introduced another description of f_i [G, p.17]. If one observes that the i -restriction gives Jordan block of L_n , this description of f_i is quite natural and not surprising at all. However, the point is that Vazirani and Grojnowski systematically developed properties of my functors and this approach is more suitable to study the modular branching rule. The crystal structure may be defined as follows.

$$\tilde{e}_i D^\lambda = \text{Soc}(e_i D^\lambda), \quad \tilde{f}_i D^\lambda = \text{Top}(f_i D^\lambda), \quad \text{wt}(D^\lambda) = \text{wt}(\lambda).$$

As the crystal we define is semiregular, ϵ_i and φ_i are determined by \tilde{e}_i and \tilde{f}_i . As is stated in the introduction, the following is proved in [G, Theorem 12.3].

Theorem 5.1. *Let $(B, \tilde{e}_i, \tilde{f}_i, \text{wt}, \epsilon_i, \varphi_i)$ be as above. Then B is isomorphic to $B(\Lambda)$.*

Another result of Grojnowski and Vazirani [GV, Lemma 3.5] implies that we can detect $\tilde{e}_i D^\lambda$ on the Grothendieck group level.

Proposition 5.1. *If $\tilde{e}_i D^\lambda \neq 0$, $\tilde{e}_i D^\lambda \neq 0$ is a unique composition factor D^μ of $e_i D^\lambda$ with $\epsilon_i(D^\mu) = \epsilon_i(D^\lambda) - 1$, and if D^ν is another composition factor then $\epsilon_i(D^\nu) < \epsilon_i(D^\mu)$.*

In the following, we denote by B the second crystal, and by \mathcal{KP} the first crystal defined on the same set B .

6. PROOF OF THE MODULAR BRANCHING RULE

We assume the conditions $q \neq 1$ and $v_i = q^{\gamma_i}$ as before.

Theorem 6.1. *For $\lambda \in \mathcal{KP}$, we have that $\tilde{e}_i D^\lambda \neq 0$ if and only if $\tilde{e}_i \lambda \neq 0$ and if this holds then $\tilde{e}_i D^\lambda = D^{\tilde{e}_i \lambda}$.*

Proof. We first assume that the characteristic of F is zero.

As \mathcal{KP} and $B = \{D^\lambda | \lambda \in \mathcal{KP}\}$ are isomorphic crystals by theorem 5.1, there exists a bijection $c : \mathcal{KP} \simeq \mathcal{KP}$ such that

$$\tilde{e}_i D^{c(\lambda)} = D^{c(\tilde{e}_i \lambda)}, \quad \tilde{f}_i D^{c(\lambda)} = D^{c(\tilde{f}_i \lambda)}, \quad \text{wt}(c(\lambda)) = \text{wt}(D^{c(\lambda)}) = \text{wt}(\lambda),$$

$$\epsilon_i(D^{c(\lambda)}) = \epsilon_i(\lambda), \quad \varphi_i(D^{c(\lambda)}) = \varphi_i(\lambda).$$

We prove by induction on n that $c(\lambda) = \lambda$ for $\lambda \vdash n$. If $n = 0$ there is nothing to prove. If $n = 1$, D^λ is the one dimensional module of the truncated polynomial ring \mathcal{H}_1 on which L_1 acts as $q^i \in \{v_1, \dots, v_m\}$ where i is the color of the unique node of λ . Thus, $\tilde{e}_i D^\lambda = D^\emptyset = D^{c(\tilde{e}_i \lambda)}$ and

$$D^{c(\tilde{e}_i c^{-1}(\lambda))} = \tilde{e}_i D^\lambda = D^{c(\tilde{e}_i \lambda)} \neq 0.$$

Then, $c(\tilde{e}_i c^{-1}(\lambda)) = c(\tilde{e}_i \lambda) \neq 0$, which implies $c(\lambda) = \lambda$.

Assume that $n > 1$ and that $c(\mu) = \mu$ for all $|\mu| < n$. Let $D^\mu = \tilde{e}_i D^\lambda \neq 0$. Then, $c(\mu) = \mu$ implies

$$\epsilon_i(c^{-1}(\lambda)) = \epsilon_i(D^\lambda) = \epsilon_i(D^\mu) + 1 = \epsilon_i(\mu) + 1,$$

$$\varphi_i(c^{-1}(\lambda)) = \varphi_i(D^\lambda) = \varphi_i(D^\mu) - 1 = \varphi_i(\mu) - 1.$$

By theorem 4.1 and lemma 3.2, we have

$$f_i P^\mu = (P^{\tilde{f}_i \mu})^{\oplus(\epsilon_i(\mu)+1)} \bigoplus \left(\bigoplus_{\lambda'} (P^{\lambda'})^{\oplus a_{\mu\lambda'}^i} \right),$$

where $a_{\mu\lambda'}^i$ are certain nonnegative integers, and λ' satisfy $\lambda' \vdash n$ and

$$\epsilon_i(\lambda') \geq \epsilon_i(\mu) + 2 > \epsilon_i(c^{-1}(\lambda)).$$

As $D^\lambda = \tilde{f}_i D^\mu = \text{Top}(f_i D^\mu)$ and we have surjection $f_i P^\mu \rightarrow f_i D^\mu$, λ is either $\tilde{f}_i \mu$ or one of λ' . If $\lambda = \tilde{f}_i \mu$ then

$$D^\mu = \tilde{e}_i D^{\tilde{f}_i \mu} = D^{c(\tilde{e}_i c^{-1}(\tilde{f}_i \mu))} = D^{\tilde{e}_i c^{-1}(\tilde{f}_i \mu)}.$$

Thus $\mu = \tilde{e}_i c^{-1}(\tilde{f}_i \mu) \neq 0$ implies $\tilde{f}_i \mu = c^{-1}(\tilde{f}_i \mu)$ and $c(\lambda) = \lambda$ follows. Hence, we may assume $\epsilon_i(\lambda) > \epsilon_i(c^{-1}(\lambda))$. Next, we consider

$$e_i P^\lambda = (P^{\tilde{e}_i \lambda})^{\oplus(\varphi_i(\lambda)+1)} \bigoplus \left(\bigoplus_{\mu'} (P^{\mu'})^{\oplus b_{\lambda\mu'}^i} \right),$$

where $b_{\lambda\mu'}^i$ are certain nonnegative integers, and μ' satisfy $\mu' \vdash n-1$ and

$$\varphi_i(\mu') \geq \varphi_i(\lambda) + 2.$$

Recall that \mathcal{H}_n is a symmetric algebra. As $D^\mu = \tilde{e}_i D^\lambda = \text{Soc}(e_i D^\lambda)$ and we have injection $e_i D^\lambda \rightarrow e_i P^\lambda$, μ is either $\tilde{e}_i \lambda$ or one of μ' . If $\mu = \tilde{e}_i \lambda$ then

$$D^\lambda = \tilde{f}_i D^{\tilde{e}_i \lambda} = D^{c(\tilde{f}_i c^{-1}(\tilde{e}_i \lambda))} = D^{c(\tilde{f}_i \tilde{e}_i \lambda)} = D^{c(\lambda)}.$$

Thus $c(\lambda) = \lambda$ again follows. Hence, we may assume $\varphi_i(\mu) \geq \varphi_i(\lambda) + 2$. As $\varphi_i(c^{-1}(\lambda)) = \varphi_i(\mu) - 1$, this implies $\varphi_i(c^{-1}(\lambda)) > \varphi_i(\lambda)$.

If both $\epsilon_i(\lambda) > \epsilon_i(c^{-1}(\lambda))$ and $\varphi_i(c^{-1}(\lambda)) > \varphi_i(\lambda)$ hold,

$$\varphi_i(c^{-1}(\lambda)) - \epsilon_i(c^{-1}(\lambda)) > \varphi_i(\lambda) - \epsilon_i(c^{-1}(\lambda)) > \varphi_i(\lambda) - \epsilon_i(\lambda).$$

Thus $wt(c^{-1}(\lambda))(h_i) > wt(\lambda)(h_i)$, which contradicts to $wt(c^{-1}(\lambda)) = wt(\lambda)$. We have proved the theorem when F is of characteristic zero.

Now we consider the positive characteristic case. Let (K, R, F) be a modular system with parameters such that the characteristic of K is zero, $\hat{q} \in R$ is a primitive e^{th} root of unity, and \hat{q} maps to $q \in F$. The image of S_R^λ in D_K^λ is denoted by D_R^λ . Since both \hat{q} and q have the multiplicative order e , we have $e_i D_K^\lambda = e_i D_R^\lambda \otimes_R K$. We also have surjection $e_i D_R^\lambda \otimes_R F \rightarrow e_i D_F^\lambda$, because e_i is exact. Since we can read $\epsilon_i(D^\lambda) = \max\{k \in \mathbb{Z}_{\geq 0} | \tilde{e}_i^k D^\lambda \neq 0\}$ from its restriction to the commutative subalgebra generated by L_1, \dots, L_n , that we have surjection $D_R^\lambda \rightarrow D_F^\lambda$ and injection $D_R^\lambda \rightarrow D_K^\lambda$ implies $\epsilon_i(D_K^\lambda) \geq \epsilon_i(D_F^\lambda)$. However, theorem 5.1 guarantees that the sum of the left hand side and the right hand side in each weight space is the same. Hence, by the proof for the characteristic zero case, we have $\epsilon_i(D_F^\lambda) = \epsilon_i(D_K^\lambda) = \epsilon_i(\lambda)$ and, by proposition 5.1, $\tilde{e}_i D_F^\lambda = \text{Soc}(e_i D_F^\lambda)$ is the composition factor D_F^μ of $e_i D_F^\lambda$ with the value $\epsilon_i(D_F^\mu) = \epsilon_i(\lambda) - 1$. Observe that

$e_i D_F^\lambda$ is self dual. Thus $\text{Top}(e_i D_F^\lambda) = D_F^\mu$. Let P_R^μ be the lift of P_F^μ . Then we have surjection $P_R^\mu \rightarrow e_i D_F^\lambda$. Consider the surjection $e_i D_R^\lambda \rightarrow e_i D_R^\lambda \otimes_R F \rightarrow e_i D_F^\lambda$. Then the surjection $P_R^\mu \rightarrow e_i D_F^\lambda$ lifts to $P_R^\mu \rightarrow e_i D_R^\lambda$, which we denote by f . Recall that P_R^μ has Specht filtration $P_R^\mu = F_0 \supset F_1 \supset \cdots$ such that $F_0/F_1 = S_R^\mu$. Let f' be the composition of f with the surjection $e_i D_R^\lambda \rightarrow \text{Top}(e_i D_F^\lambda)$. As f' factors through $P_F^\mu = P_R^\mu \otimes_R F$ and $F_1 \otimes_R F$ is a proper submodule of $P_F^\mu = F_0 \otimes_R F$ because $F_0 \otimes_R F/F_1 \otimes_R F = S_F^\mu$, that P_F^μ is the projective cover of $\text{Top}(e_i D_F^\lambda)$ implies that $f'(F_1) = 0 \subset \text{Top}(e_i D_F^\lambda) = f'(F_0)$. We have proved $f(F_0) \neq f(F_1)$. Let $K = \text{Ker } f$. Then we have

$$0 \rightarrow K \rightarrow F_0 \rightarrow f(F_0) \rightarrow 0.$$

Since these are free R -modules, the exact sequence splits as R -modules. Thus, there exists a surjective R -linear map

$$f(F_0)/f(F_1) \bigoplus K/K \cap F_1 \rightarrow F_0/F_1 = S_R^\mu.$$

Suppose that $f(F_0)$ and $f(F_1)$ have the same rank as R -modules. Then $K/K \cap F_1 \rightarrow F_0/F_1$ is surjective, since $f(F_0)/f(F_1)$ is a torsion R -module and S_R^μ is a free R -module. Thus, for any $x \in F_0$, we may write $x = y + z$ where $y \in F_1$ and $z \in K$, which implies $f(x) = f(y) \in f(F_1)$ and $f(F_0) = f(F_1)$, which is a contradiction. Therefore, we must have $f(F_0) \otimes_R K \neq f(F_1) \otimes_R K$. Consider the surjection $S_K^\mu \rightarrow f(F_0) \otimes_R K/f(F_1) \otimes_R K$. Since μ is Kleshchev and since $f(F_0) \otimes_R K/f(F_1) \otimes_R K \neq 0$, the kernel of the map is contained in $\text{Rad } S_K^\mu$. Thus we have $[f(F_0) \otimes_R K/f(F_1) \otimes_R K : D_K^\mu] \neq 0$. As $f(F_0) \otimes_R K/f(F_1) \otimes_R K$ is a subquotient of $e_i D_R^\lambda \otimes_R K$, D_K^μ appears as a composition factor of $e_i D_K^\lambda$ with $\epsilon_i(D_K^\mu) \geq \epsilon_i(\lambda) - 1$. As the maximum value in $e_i D_K^\lambda$ is $\epsilon_i(\lambda) - 1$ and it is attained by $D_K^{\tilde{e}_i \lambda}$ by the proof in the characteristic zero case, we conclude that $\mu = \tilde{e}_i \lambda$ as desired. \square

Remark 6.2. As a corollary, $\dim D^\lambda$ is greater than or equal to the number of paths from \emptyset to λ in \mathcal{KP} .

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